1. Show that \( \{ |+\rangle := \left( \frac{1}{\sqrt{2}} \right), |-\rangle := \left( \frac{1}{\sqrt{2}} \right) \} \) forms an orthonormal basis for \( \mathbb{C}^2 \) (i.e. show that the two vectors are orthogonal and that they have norm 1).

**Solution:**
\[
|||+\rangle||^2 = |\langle + | + \rangle| = (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1
\]
\[
|||-\rangle||^2 = |\langle - | - \rangle| = (\frac{1}{\sqrt{2}})^2 + (-\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1
\]
\[
\langle + | - \rangle = \frac{1}{2} - \frac{1}{2} = 0
\]

2. In this exercise you will show how to convert a state from the computational basis \( \{ |0\rangle, |1\rangle \} \) to the Hadamard basis \( \{ |+\rangle, |-\rangle \} \).

(a) Write the vector \(|0\rangle\) in the Hadamard basis (i.e. find the values \(\alpha\) and \(\beta\) such that \(|0\rangle = \alpha |+\rangle + \beta |-\rangle\)). Do the same thing for \(|1\rangle\).

**Solution:**
\[
|0\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle
\]
\[
|1\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle
\]

(b) Let \(|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle\). Write \(|\psi\rangle\) in the Hadamard basis.

**Solution:**
\[
|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle
\]
\[
= \alpha_0 \left( \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \right) + \alpha_1 \left( |1\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \right)
\]
\[
= \alpha_0 + \alpha_1 \left( \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \right).
\]

3. Let \(|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{6}} |11\rangle\), \(|\phi\rangle = |10\rangle\), \(|\rho\rangle = \frac{1}{\sqrt{2}} (|00\rangle - i |11\rangle)\) and \(|\gamma\rangle = \frac{1}{\sqrt{2}} (i |00\rangle + |11\rangle)\). Compute

(a) \(\langle \psi |, \langle \phi |, \langle \rho |\)

**Solution:**
\[
\langle \psi | = \frac{1}{\sqrt{2}} \langle 00 | + \frac{1}{\sqrt{3}} \langle 01 | + \frac{1}{\sqrt{6}} \langle 11 | = \left( \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \frac{1}{\sqrt{6}} \right)
\]
\[
\langle \phi | = \langle 10 | = (0 \quad 0 \quad 1 \quad 0)
\]
\[
\langle \rho | = \frac{1}{\sqrt{2}} \langle 00 | + i \langle 11 | = \left( \frac{1}{\sqrt{2}} 0 0 \frac{i}{\sqrt{2}} \right)
\]

(b) \(\langle \psi | \phi \rangle, \langle \psi | \rho \rangle, \langle \gamma | \rho \rangle, \langle \rho | \gamma \rangle\)

**Solution:**
\[
\langle \psi | \phi \rangle = 0
\]
\[
\langle \psi | \rho \rangle = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{12}}
\]
\[
\langle \gamma | \rho \rangle = \frac{1}{2} + \frac{3}{2} = -i
\]
\[
\langle \rho | \gamma \rangle = \frac{1}{2} + \frac{1}{2} = i
\]

**Remark:** For every \(|\psi\rangle, |\phi\rangle\), \(\langle \psi | \phi \rangle = \langle \phi | \psi \rangle\) (exercise!)
(c) $|\psi\rangle\langle\psi|, |\phi\rangle\langle\rho|, |\rho\rangle\langle\rho|, |\gamma\rangle\langle\gamma|$

**Solution:**

$$
|\psi\rangle\langle\psi| = \begin{pmatrix}
\frac{1}{2} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{18}} \\
\frac{1}{\sqrt{6}} & \frac{1}{3} & 0 & \frac{1}{\sqrt{18}} \\
0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} & 0 & \frac{1}{6}
\end{pmatrix}
$$

$$
|\phi\rangle\langle\rho| = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
|\rho\rangle\langle\rho| = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{i}{2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{i}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix}
$$

$$
|\gamma\rangle\langle\gamma| = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{i}{2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{i}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix}
$$

**Remark:** For every $|\psi\rangle$ and $|\phi\rangle = e^{i\theta} |\psi\rangle$, $|\psi\rangle\langle\psi| = |\phi\rangle\langle\phi|$ (exercise!)

4. Show that for every quantum state $|\psi\rangle$:

(a) $\langle\psi|\psi\rangle = 1$;

**Solution:**

Let $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_d \end{pmatrix}$. We have that $\langle\psi| = (\alpha_1 \cdots \alpha_d)$. 

$$
\langle\psi|\psi\rangle = \sum_i \alpha_i^* \alpha_i = \sum_i |\alpha_i|^2 = 1, \text{ where the last equality holds since } |\psi\rangle \text{ is a quantum state.}
$$

(b) $|\psi\rangle\langle\psi|$ is a projector ($P$ is a projector if $P^2 = P$).

**Solution:**

$$
|\psi\rangle\langle\psi| |\psi\rangle\langle\psi| = \langle\psi|\psi\rangle |\psi\rangle\langle\psi| = 1 \cdot |\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|
$$

5. Find the matrix that represents the unitaries

$$
Z |b\rangle = (-1)^b |b\rangle \\
P |b\rangle = i^b |b\rangle \\
c(X) |a\rangle |b\rangle = |a\rangle |a \oplus b\rangle
$$

where $b \in \{0, 1\}$ and $\oplus$ denotes the XOR operation.

**Solution:**

$$
Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\
c(X) = CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

6. Show that if $U_1$ and $U_2$ are unitaries then $U_1U_2$, $U_1 \otimes U_2$ and $c(U_1)$ are unitaries.

**Solution:**

$$
U_1U_2(U_1U_2)^\dagger = U_1U_2U_2^\dagger U_1^\dagger = U_1U_1^\dagger = I
$$

$$
(U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger = (U_1 \otimes U_2)(U_1^\dagger \otimes U_2^\dagger) = (U_1U_1^\dagger \otimes U_2U_2^\dagger) = I
$$
7. Let \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \). Compute \( H |\psi\rangle \) (i.e. find the values of \( \alpha' \) and \( \beta' \) such that \( H |\psi\rangle = \alpha' |0\rangle + \beta' |1\rangle \)).

\[
|\psi\rangle = \frac{\alpha |0\rangle + \beta |1\rangle}{\sqrt{2}}
\]

\[
H |\psi\rangle = \frac{\alpha \sqrt{2} |0\rangle + \beta \sqrt{2} |1\rangle}{2}
\]

8. Find the outcomes and the corresponding probabilities and post-measurement states of the measurement of both qubits of the state \( \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) in the computational basis. Do the same for the measurement of both qubits in the Hadamard basis.

**Solution:**

**Computational basis:**

- 00 w.p. \( \frac{1}{\sqrt{2}} \) and post-measurement state is \( \frac{1}{\sqrt{2}} |00\rangle = |00\rangle \)
- 01 and 10 w.p. 0
- 11 w.p. \( \frac{1}{\sqrt{2}} \) and post-measurement state is \( \frac{1}{\sqrt{2}} |11\rangle = |11\rangle \)

**Hadamard basis:** The state written in the Hadamard basis is

\[
\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)
\]

\[
= \frac{1}{2 \sqrt{2}} \left( \left( \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \right) \left( \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \right) + \left( \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \right) \left( \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \right) \right)
\]

\[
= \frac{1}{2 \sqrt{2}} |++\rangle + \frac{1}{2 \sqrt{2}} |+-\rangle + \frac{1}{2 \sqrt{2}} |-+\rangle + \frac{1}{2 \sqrt{2}} |--\rangle
\]

\[
+ \frac{1}{2 \sqrt{2}} |++\rangle + \frac{1}{2 \sqrt{2}} |+-\rangle - \frac{1}{2 \sqrt{2}} |-+\rangle - \frac{1}{2 \sqrt{2}} |--\rangle
\]

\[
= \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle)
\]

The measurement outcomes are:

- ++ w.p. \( \frac{1}{\sqrt{2}} \) and post-measurement state is \( \frac{1}{\sqrt{2}} |++\rangle = |++\rangle \)
- +– and –+ w.p. 0
- -- w.p. \( \frac{1}{\sqrt{2}} \) and post-measurement state is \( \frac{1}{\sqrt{2}} |--\rangle = |--\rangle \)

9. Let \( |\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{6}} |11\rangle \). We measured the second qubit of \( |\psi\rangle \) in the computational basis and the outcome was 1. Then, we measured the first qubit (of the state after the first measurement) in the Hadamard basis. Compute the outcomes and corresponding probabilities and post-measurement states of the second measurement.
Solution:
The first measurement has outcomes:

0 w.p. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and the post-measurement state is $|00\rangle$

1 w.p. $|\frac{1}{\sqrt{3}}|^2 + |\frac{1}{\sqrt{6}}|^2 = \frac{1}{2}$ and the post-measurement state is $\sqrt{2} \cdot (\frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{6}} |11\rangle) = \frac{\sqrt{2}}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle$

Since we assume that the outcome of the first measurement is 1, we have the state $\frac{\sqrt{2}}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle$. If we write the first qubit in the Hadamard basis, we have

$$\frac{\sqrt{2}}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle = \frac{1}{\sqrt{3}}(|+,1\rangle + |-,1\rangle) + \frac{1}{\sqrt{6}}(|+,1\rangle - |-,1\rangle)$$

$$= \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right) |+,1\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right) |-,1\rangle$$

The outcome of measuring the first qubit in the Hadamard basis is:

$+$ with probability $|\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|^2 = \frac{1}{2} + \frac{1}{3\sqrt{2}}$ and the post-measurement state is $|+1\rangle$

$-$ with probability $|\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|^2 = \frac{1}{2} - \frac{1}{3\sqrt{2}}$ and the post-measurement state is $|-1\rangle$